Basic Complex Analysis Marsden Hoffman

1.1-1.2: Introduction to the Complex Plane

lecture 1 topics

complex numbers and complex plane $\mathbb{C}$
addition, multiplication, conjugation, division.
polar coordinates for complex numbers
addition and multiplication interpreted geometrically.

Complex analysis is like Calculus and analysis that you've studied in previous courses - it's based on derivatives and integrals and analysis concepts - except that the functions $f(z)$ have complex number domains and ranges, i.e. domain and range are subsets of the *complex plane*. $f'(z)$, $\int f(z)dz$ magic

- There is an overview of complex analysis at Wikipedia, if you're interested.

The starting point of complex analysis is to understand the *complex plane* $\mathbb{C}$.

You may or may not have discussed the geometry of $\mathbb{C}$ in a linear algebra course or elsewhere; under addition and real scalar multiplication $\mathbb{C}$ is isomorphic to the real vector space $\mathbb{R}^2$. General multiplication in $\mathbb{C}$ is more interesting geometrically and we'll understand it in this lecture.

Definition The *complex plane* $\mathbb{C}$ is defined as a set by

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}.$$

If $z = x + i\,y$ with $x, y \in \mathbb{R}$ then the *real part* of z, $\mathrm{Re}(z)$ is x; and the *imaginary part of* z, $\mathrm{Im}(z)$ is y.

is a real#!

$\mathrm{Re}(3+4i) = 3$

$\mathrm{Im}(3+4i) = 4$

Two complex numbers are *equal* if and only their real parts and imaginary parts are equal.

$\mathbb{C}$ is an *algebra* under the operations of addition and multiplication, as defined by

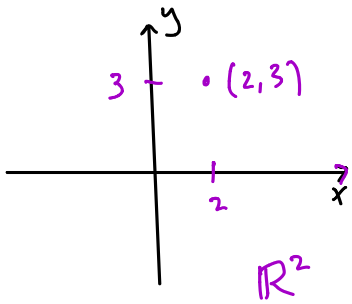
- $$\overset{z}{(x_1 + i\,y_1)} + \overset{w}{(x_2 + i\,y_2)} := \overset{\mathrm{Re}(z+w)}{(x_1 + x_2)} + \overset{\mathrm{Im}(z+w)}{i(y_1 + y_2)}$$
- $$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$

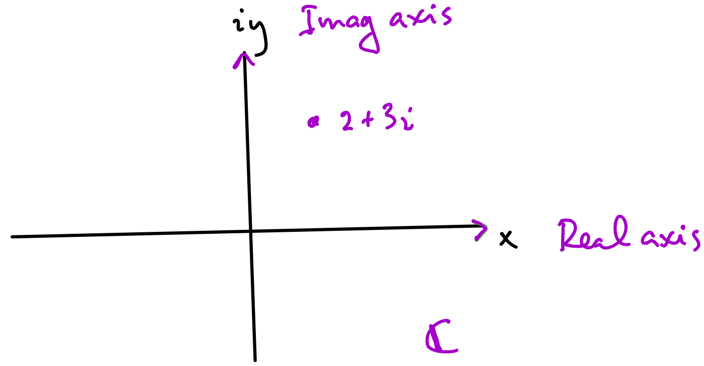
for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

what you get if you "foil" LHS, replace $i^2 y_1y_2$ with $-y_1y_2$

Note that the definition for complex multiplication is equivalent to using the usual axioms for real number multiplication and addition, together with the introduction of a symbol i which has the property that $i^2 := -1$.

It is natural to identify each complex number $x + i\,y \in \mathbb{C}$ with the corresponding point $(x, y) \in \mathbb{R}^2$. This identification and the usual representation for $\mathbb{R}^2$ is how we study the *complex plane* $\mathbb{C}$ as a geometric object.

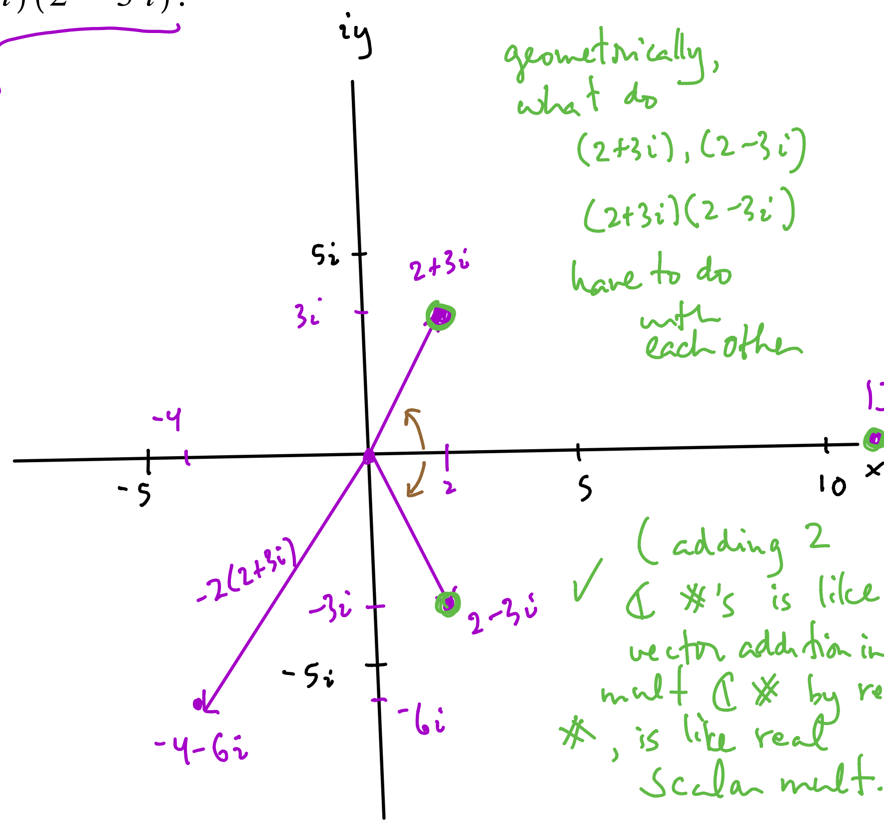




Example 1: Use the identification of $\mathbb{C}$ with $\mathbb{R}^2$ to sketch some points in the complex plane and add and multiply some complex numbers e.g.

$2 + 3\,i,\ -2(2 + 3\,i),\ 2 - 3\,i,\ \underbrace{(2 + 3\,i)(2 - 3\,i)}_{=13}.$

$-4-6i$
like real scalar mult



geometrically, what do $(2+3i)$, $(2-3i)$, $(2+3i)(2-3i)$ have to do with each other

✓ (adding 2 $\mathbb{C}$ #'s is like vector addition in $\mathbb{R}^2$
mult $\mathbb{C}$ # by real #, is like real scalar mult.

As well as adding, subtracting and multiplying complex numbers, you can also divide two complex numbers by each other, i.e. $\mathbb{C}$ satisfies the axioms of a *field*: Precisely, let $\boldsymbol{z}, w, s \in \mathbb{C}$. Then we have the following properties:

Field axioms:

the addition axioms, which correspond to vector space axioms in $\mathbb{R}^2$:

$$\boldsymbol{z} + w = w + \boldsymbol{z}$$

$$\boldsymbol{z} + (w + s) = (\boldsymbol{z} + w) + s$$

$$\boldsymbol{z} + 0 = \boldsymbol{z}$$

$$\boldsymbol{z} + (-1 \cdot \boldsymbol{z}) = 0;$$

the multiplication axioms, which are straightforward and which you will check in homework:

$$\boldsymbol{z}\, w = w\, \boldsymbol{z}$$

$$(\boldsymbol{z}\, w)\, s = \boldsymbol{z}(w\, s)$$

$$1\, \boldsymbol{z} = \boldsymbol{z}\ ;$$

distributive property of multiplication over addition.

$$\boldsymbol{z}(w + s) = \boldsymbol{z}\, w + \boldsymbol{z}\, s;$$

and finally

each $\boldsymbol{z} \neq 0$ has unique $\boldsymbol{z}^{-1} \in \mathbb{C}$ which we write as $\frac{1}{\boldsymbol{z}}$, such that $\boldsymbol{z}\,\boldsymbol{z}^{-1} = 1$

and $\frac{w}{\boldsymbol{z}} := w\,\boldsymbol{z}^{-1}$ (in particular $\frac{\boldsymbol{z}}{\boldsymbol{z}} = 1$).

Example 2 Verify that

$$\frac{1}{2+3\,i} = \frac{1}{13}(2-3\,i) = \frac{2}{13} - \frac{3}{13}i$$

How did I know that???

Complex conjugation

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then the *complex conjugate* of z, also called *z bar* and written as $\overline{z}$ is defined to be

$$\overline{z} := x - i\,y$$

And the *modulus* or *absolute value* of z is defined to be

$$|\,z\,| := \sqrt{x^2 + y^2}$$

Check:

$\overline{zw} = \overline{z}\,\overline{w}$.

$|\,z\,|^2 = z\,\overline{z}$ so $|\,z\,| = \sqrt{z\,\overline{z}}$.

and so the absolute value of a product is the product of the absolute values:

$|\,z\,w\,| = |\,z\,|\,|w|$

and this is how you compute reciprocols:

$$\frac{1}{z} = \frac{1}{z}\,\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|\,z\,|^2}$$

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}.$$
$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2\,) + i(y_1 + y_2)$$
$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$
$$\text{for all } x_1, y_1, x_2, y_2 \in \mathbb{R}.$$

Under the identification of $\mathbb{C}$ with $\mathbb{R}^2$, the definition for complex number addition just corresponds to vector addition in $\mathbb{R}^2$ (considered as a vector space), which we understand and as we illustrated in the previous example. The product of a real number with a complex number corresponds to scalar multiplication in $\mathbb{R}^2$, which we also understand geometrically.

$$\begin{array}{ll} \mathbb{C}: & (x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2\,) + i(y_1 + y_2) \\ \mathbb{R}^2: & (x_1\,,\ y_1) + (x_2\,, y_2) := (x_1 + x_2\,, y_1 + y_2) \end{array}$$

$$\begin{array}{ll} \mathbb{C}: & x_1(x_2 + i\,y_2) := x_1x_2 + i\,x_1\,y_2 \\ \mathbb{R}^2: & x_1(x_2\,, y_2) := (x_1\,x_2\,, x_1\,y_2) \end{array}$$

The more general formula for complex multiplication has geometric meaning. This magic meaning is not immediately apparent using Cartesian coordinates, as the formula in $\mathbb{R}^2$ looks sort of mysterious. But polar coordinates will solve the mystery.

$$\begin{array}{ll} \mathbb{C}: & (x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\ \mathbb{R}^2: & (x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2\,,\ x_1y_2 + y_1x_2)\,. \end{array}$$

Polar form of complex numbers and the geometric meaning of complex multiplication.

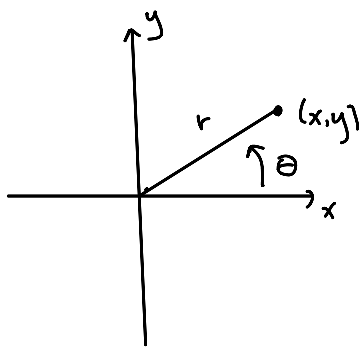
Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

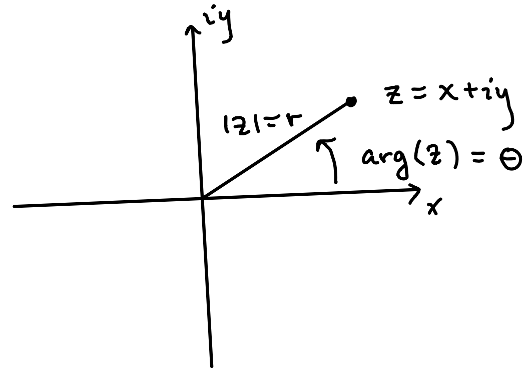
$$(x, y) = (r\cos\theta\,,\, r\sin\theta\,) = r(\cos\theta, \sin\theta)$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle in radians from the positive $x-$axis to the point (x, y), determined up to an integer multiple of $2\,\pi$. In complex form this reads

$$\boldsymbol{z} = x + i\,y = \; r(\cos\theta + \,i\,\sin\theta).$$

Note that $r = |\,\boldsymbol{z}\,|$ is the absolute value of $\boldsymbol{z}$, using complex notation. And we also have a special name for the polar angle θ, we call it the *argument of* $\boldsymbol{z}$, or $arg(\boldsymbol{z})$ for short.

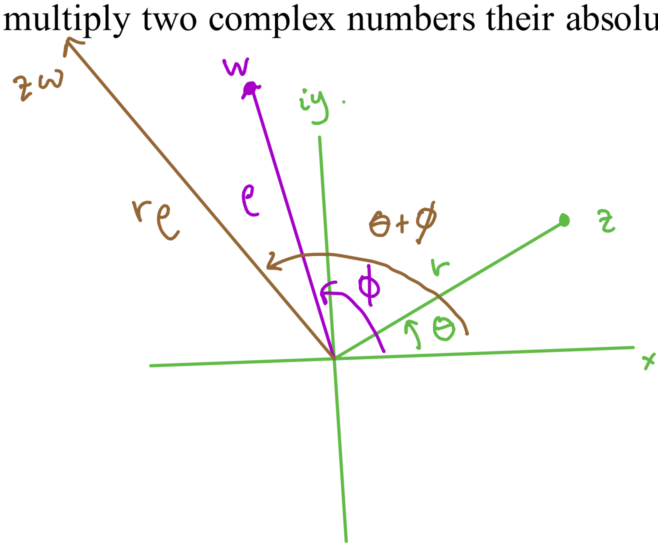




<u>Theorem</u>: Let $\boldsymbol{z} = \; r(\cos\theta + \,i\,\sin\theta)$ and $w = \rho(\cos\phi\, + \,i\,\sin\phi\,)$ be complex numbers written in polar form. Then

$$\boldsymbol{z}\,w = r\,\rho\,\left(\cos(\theta+\phi) + \,i\,\sin(\theta+\phi)\right).$$

In other words, when you multiply two complex numbers their absolute values multiply and their arguments add!



Note: If you use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ from Math 2280 ,then the multiplication formula from the previous page is particularly nice and concise: Let

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta}$$

$$w = |w|(\cos(\phi + i\sin(\phi)) = |w|e^{i\phi},$$

then product

$$zw = |z|e^{i\theta}\,|w|e^{i\phi} = |z|\,|w|e^{i\cdot(\theta+\phi)}$$

Example 4 Express $z = 1 + i$ in polar form. Compute z^2, z^3, $\frac{1}{z}$ using rectangular and polar form. Sketch!! To be continued!